Exercise 5

(a) Using the solution in Exercise 3, solve the wave equation with initial data

$$u(x,0) = \frac{1}{1+x^2}, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad -\infty < x < \infty.$$

(b) Take $c = 1$, and plot the solution for $-15 < x < 15$ and $t = 0, 1, 2, 4, 6, 8$. Observe that half the wave moves to the left and the other half moves to the right.

Solution

The aim is to solve the wave equation on the whole line for all time subject to initial conditions.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ -\infty < t < \infty$$

$$u(x,0) = f(x) = \frac{1}{1+x^2}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = 0$$

Start with the general solution of the wave equation.

$$u(x,t) = F(x+ct) + G(x-ct)$$

Differentiate it with respect to t.

$$\frac{\partial u}{\partial t} = F'(x+ct)\cdot\frac{\partial}{\partial t}(x+ct) + G'(x-ct)\cdot\frac{\partial}{\partial t}(x-ct) = F'(x+ct)\cdot(c) + G'(x-ct)(-c) = cF'(x+ct) - cG'(x-ct)$$

Now apply the given initial conditions.

$$u(x,0) = F(x) + G(x) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = cF'(x) - cG'(x) = g(x)$$

This is a system of two equations with two unknowns, F and G, that can be solved for. Differentiate both sides of the first equation.

$$\begin{cases} F'(x) + G'(x) = f'(x) \\ cF'(x) - cG'(x) = g(x) \end{cases}$$

Multiply both sides of the first equation by c.

$$\begin{cases} cF'(x) + cG'(x) = cf'(x) \\ cF'(x) - cG'(x) = g(x) \end{cases}$$

Adding the respective sides of these equations eliminates G and gives

$$2cF'(x) = cf'(x) + g(x).$$

Divide both sides by $2c$.

$$F'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x)$$

Integrate both sides with respect to x.

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int^{x} g(s)\,ds + C_1$$

Subtracting the respective sides of these equations instead eliminates F and gives

$$2cG'(x) = cf'(x) - g(x).$$

Divide both sides by $2c$.

$$G'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x)$$

Integrate both sides with respect to x.

$$\begin{aligned} G(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int^{x} g(s)\,ds + C_2 \\ &= \frac{1}{2}f(x) + \frac{1}{2c}\int_{x} g(s)\,ds + C_2 \end{aligned}$$

Now that F and G are known, the solution to the initial value problem can be written.

$$\begin{aligned} u(x,t) &= F(x+ct) + G(x-ct) \\ &= \left[\frac{1}{2}f(x+ct) + \frac{1}{2c}\int^{x+ct} g(s)\,ds + C_1\right] + \left[\frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct} g(s)\,ds + C_2\right] \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds + C_3 \end{aligned}$$

The integration constant is set to zero to satisfy $u(x,0) = f(x)$.

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

In this exercise

$$f(x) = \frac{1}{1+x^2} \quad \text{and} \quad g(x) = 0,$$

so

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} (0)\,ds \\ &= \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right]. \end{aligned}$$

Below is a plot of $u(x, t)$ versus x over $-15 < x < 15$ for $t = 0, 1, 2, 4, 6, 8$ with $c = 1$.